International Journal of Theoretical Physics, Vol. 29, No. 1, 1990

Algebraically Special Spacetimes

Garry Ludwig[1]

Received May 5, 1989

The Newman-Penrose formalism for algebraically special spacetimes, with or without twist, is recast in terms of weighted quantities defined at conformal null infinity J^+. Weighted differential operators, also defined at J^+, are introduced that are special cases of those defined in a recent extension of the Geroch-Held-Penrose formalism. The "solution," including spin coefficients, Weyl tensor components, and reduced equations, is expressed rather concisely in terms of these weighted variables and operators. Its form invariance under the remaining freedom in the choice of tetrad and coordinate system now becomes evident.

1. INTRODUCTION

Partial "solutions" to Einstein's field equations and corresponding reduced gravitational field equations (that is, the equations yet to be solved after certain radial integrations have been made) were obtained for algebraically special spacetimes with twisting rays about two decades ago by various authors. For vacuum they were derived by Kerr (1963), Debney *et al.* (1969), Robinson *et al.* (1969), and Talbot (1969). Their results were generalized by Lind (1974) and Trim and Wainwright (1974) to the nonvacuum case, but only for a certain class of Ricci tensors. I considered a slightly bigger class of Ricci tensors (Ludwig, 1978), but the "solution" reduced to that of Trim and Wainwright (1974).

The "solution" obtained, including the metric, spin coefficients, and components of the Weyl and Ricci tensors, is a fairly messy combination of integration "constants" and their nonradial derivatives. By just looking at the results, their invariance under the remaining freedom in the choice of frame, i.e., under a certain combined transformation of the spinor dyad (or, equivalently, the tetrad), the coordinate system, and the initial data is far from obvious. I intend to reformulate these results in terms of properly weighted scalar functions and differential operators defined at conformal

[1]Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

0020-7748/90/0100-0045$06.00/0

null infinity. In particular, the differential operators, when acting on properly weighted scalar functions, transform as properly weighted quantities under arbitrary diagonal transformations (Ludwig, 1986, 1988) of the spinor dyad, transformations which include both Lorentz and conformal ones. These operators are actually special cases of those defined in a recent extension of the Geroch–Held–Penrose formalism (Ludwig, 1988). The present reformulation makes it possible to exhibit the "solution" not only in a very concise form, but also in a way that makes its variance under the remaining choice of frame self-evident.

The "solution" was obtained in Ludwig (1978) with the aid of Penrose's conformal technique (Penrose, 1968). By rescaling the metric, the spacetime (M, g_{ab}) was transformed, subject to a number of assumptions, to an unphysical space $(\hat{M}, \hat{g}_{ab})$ with boundary (a local J^+). The frame, including spinor dyad, coordinate system, and conformal factor, was chosen at J^+ at the outset. The Newman–Penrose equations (Newman and Penrose, 1962; Pirani, 1965) were then integrated along ingoing shear-free null geodesics and the results converted back to physical spacetime. All integration "constants" appeared as quantities defined on J^+. By using the remaining freedom in the choice of frame, the twist Σ, assumed to be nonzero, was put equal to unity. In the present reformulation I reverse this choice and let Σ have arbitrary values, including zero.

The notation follows that of previous papers (Ludwig, 1978; 1986). Careted quantities refer to the rescaled space $\hat{M}$, uncareted ones refer to spacetime M. Superscripts on a careted variable denote the appropriate coefficient in the expansion of that variable in powers of the conformal factor Ω; superscripts on an uncareted variable similarly refer to the expansion of that variable in powers of ρ and $\bar{\rho}$, where ρ is a suitable spin-coefficient. The usual symbols are used for the Newman–Penrose quantities (Newman and Penrose, 1962; Pirani, 1965). Their transformation properties under general Lorentz and conformal transformations are found in the literature (Ludwig, 1986).

2. THE CHOICE OF FRAME

The spinor dyad and conformal factor can be so chosen on J^+ such that

$$\hat{\lambda}^0 = \hat{\nu}^0 = 0, \qquad \hat{\alpha}^0 = -\hat{\bar{\beta}}^0 = \tfrac{1}{2}\hat{\bar{\delta}}^0 \ln P, \qquad \hat{\gamma}^0 = -\tfrac{1}{2}\hat{\mu}^0 = \tfrac{1}{2}\hat{\Delta}^0 \ln P \tag{1}$$

and

$$\hat{\tau}^0 = 0 \tag{2}$$

where P is any positive function on J^+. Such a Type I frame (Ludwig, 1976, 1978), which includes an associated Bondi coordinate system, is, however, inappropriate for considerations of twist.

More appropriate is a Type II frame, whose construction I now review. Starting with any Type I frame, we first perform a null rotation

$$o'_A = o_A - L\iota_A, \qquad \iota'_A = \iota_A \tag{3}$$

with L so chosen that $\hat{\sigma}^{0'}$ vanishes. The relations given by (1) are still valid for the new dyad, but

$$\hat{\tau}^{0'} = P\hat{\Delta}^0\left(\frac{L}{P}\right) \tag{4}$$

Dropping all primes from now on, consider the congruence of null geodesics whose tangent vector at J^+ is

$$\hat{k}_a = \hat{o}_A\hat{\bar{o}}_{\dot{A}} \tag{5}$$

This congruence is clearly asymptotically shear free.

Next, extend the Bondi coordinate system $(u, \zeta, \bar{\zeta})$ of J^+ associated with the Type I frame to all of $\hat{M}$ by demanding that these coordinates not change along the members of the null geodesic congruence. For the remaining coordinate take the conformal factor Ω, which is so chosen that

$$\operatorname{Re}\hat{\rho} = 0 \tag{6}$$

The twist Σ of the congruence is $-\operatorname{Im}\hat{\rho}$. We propagate the tetrad into the interior of $\hat{M}$ in such a way that

$$\hat{\kappa} = \hat{\varepsilon} = \hat{\pi} + \hat{\bar{\tau}} = 0 \tag{7}$$

are satisfied identically. In the physical spacetime M we use, instead, coordinates $(u, r, \zeta, \bar{\zeta})$, where r is related to Ω by the equation

$$\Omega^2 = \frac{1}{r^2 + \Sigma^2} \tag{8}$$

In Ludwig (1978), I "solved" the Newman–Penrose equations in such a Type II frame under the assumption that the above null geodesic congruence is not only asymptotically shear free, but shear free throughout the space, i.e.,

$$\hat{\sigma} = 0 \tag{9}$$

identically. It was further assumed that the tangent vectors to these null geodesics are repeated principal null vectors of the Weyl tensor, i.e.,

$$\Psi_0 = \Psi_1 = 0 \tag{10}$$

and that the Ricci tensor satisfies

$$\Lambda = \Phi_{00} = 0 \tag{11}$$

The solution obtained depends on the initial data

$$P, L, \operatorname{Re} \hat{\Psi}_2^{(1)}, \Phi_{11}^{(4)} \tag{12}$$

The freedom in the choice of the original Type I frame is the (generalized) Newman–Unti (Ludwig, 1976, 1978) freedom. When translated to a Type II frame this becomes, in the unphysical space $\hat{M}$, a coordinate change

$$u' = G(u, \zeta, \bar{\zeta}), \qquad \zeta' = \zeta'(\zeta), \qquad \bar{\zeta}' = \bar{\zeta}'(\bar{\zeta}), \qquad \Omega' = \Theta\Omega \tag{13a}$$

where

$$\Theta \equiv \dot{G} \, (\equiv G_{,u}) \tag{13b}$$

accompanied by the diagonal transformation

$$\operatorname{diag}(\Theta^{-1/2} e^{i\phi}, \Theta^{-1/2} e^{-i\phi}) \tag{13c}$$

of the spinor dyad, where

$$e^{4i\phi} = \frac{d\bar{\zeta}'}{d\bar{\zeta}} \bigg/ \frac{d\zeta'}{d\zeta} \tag{14}$$

The corresponding freedom in the physical spacetime is given by

$$u' = G(u, \zeta, \bar{\zeta}), \qquad \zeta' = \zeta'(\zeta), \qquad \bar{\zeta}' = \bar{\zeta}'(\bar{\zeta}), \qquad r' = \Theta^{-1} r \tag{15a}$$

for the coordinate system and the transformation

$$\operatorname{diag}(\Theta^{1/2} e^{i\phi}, \Theta^{-1/2} e^{-i\phi}) \tag{15b}$$

for the spinor dyad. The initial data must at the same time be changed according to

$$\begin{aligned} P' &= P\Theta^{-1} \left| \frac{d\zeta'}{d\zeta} \right| \\ L' &= \left(L + 2P\Theta^{-1} \frac{\partial G}{\partial \zeta} \right) e^{2i\phi} \\ \operatorname{Re} \hat{\Psi}_2^{(1)\prime} &= \Theta^{-3} \operatorname{Re} \hat{\Psi}_2^{(1)} \\ \Phi_{11}^{(4)\prime} &= \Theta^{-4} \Phi_{11}^{(4)} \end{aligned} \tag{16}$$

3. WEIGHTED QUANTITIES

If under the permissible change of frame given by equations (13)–(16) a quantity $\eta(u, \zeta, \bar{\zeta})$ transforms as

$$\eta' = \Theta^W e^{2iS\phi} \eta \tag{17}$$

we say that η is a properly weighted quantity with weights (W, S). Of course, the conjugate quantity then has weights $(W, -S)$.

Next, let us define, on J^+, properly weighted differential operators $\hat{\delta}_c^0, \hat{\bar{\delta}}_c^0, \hat{\Delta}_c^0$ by

$$\begin{aligned} \hat{\delta}_c^0 &= \hat{\delta}^0 + 2S\hat{\alpha}^0 + (W-S)\hat{\tau}^0 \\ \hat{\bar{\delta}}_c^0 &= \hat{\bar{\delta}}^0 - 2S\hat{\bar{\alpha}}^0 + (W+S)\hat{\bar{\tau}}^0 \\ \hat{\Delta}_c^0 &= \hat{\Delta}^0 - W\hat{\mu}^0 \end{aligned} \tag{18}$$

They have respective weights $(-1, 1)$, $(-1, -1)$, and $(-1, 0)$. Thus, for example, if η is a scalar field with weights (W, S), then $\hat{\delta}_c^0\eta$ transforms as a weighted function with weights $(W-1, S+1)$.

The commutators of these operators can be worked out directly. They are given by

$$\begin{aligned} (\hat{\delta}_c^0\hat{\bar{\delta}}_c^0 - \hat{\bar{\delta}}_c^0\hat{\delta}_c^0)\eta &= -2i\Sigma\hat{\Delta}_c^0\eta + \eta[2SU^0 + 2iW\hat{\Delta}_c^0\Sigma] \\ (\hat{\Delta}_c^0\hat{\delta}_c^0 - \hat{\delta}_c^0\hat{\Delta}_c^0)\eta &= (W-S)\bar{\nu}^0\eta \end{aligned} \tag{19}$$

(and the equation conjugate to the latter), with ν^0, U^0, and Σ as defined in the next section.

The transformations of the spinor dyads, in both M and $\hat{M}$, as given by equations (13c) and (15b), are special cases of a general diagonal transformation diag(a, d), whose consequences were discussed in depth in a recent extension (Ludwig, 1988) of the Geroch-Held-Penrose (GHP) formalism. For weighted quantities of either space, M or $\hat{M}$, we can express the weights W and S in terms of the more general weights defined by

$$\eta' = a^r\bar{a}^s d^t\bar{d}^u\eta \tag{20}$$

for an arbitrary diagonal transformation. The present formalism is a specialization of the extended GHP formalism of Ludwig (1988), albeit a different specialization for quantities defined on $\hat{M}$ and quantities defined on M.

With the aid of equation (13c), we see that for weighted quantities defined on $\hat{M}$ we have

$$\begin{aligned} W &= -\tfrac{1}{2}(r+s+t+u) \\ S &= \tfrac{1}{2}(r-s-t+u) \end{aligned} \tag{21}$$

i.e., W is the *conformal weight* and S is the *spin weight* (Ludwig, 1988). The operators of equation (18) are those of Ludwig (1988) when specialized to the particular diagonal transformation given by equation (13c). This can be readily established with the aid of equations (1), (7), and (21). The commutator equations, too, equations (19), follow from the more general ones given in that paper.

For weighted quantities defined in the physical spacetime M the weights (W, S) are given by

$$W = \tfrac{1}{2}(r + s - t - u)$$
$$S = \tfrac{1}{2}(r - s - t + u) \tag{22}$$

as is readily seen with the aid of equation (15b). Thus, W is the *boost weight* and S the *spin weight.* In fact, the general formalism of Ludwig (1988) applied to quantities defined in M reduces to a special case of the GHP (Geroch *et al.*, 1973) formalism. It is a special case because the boost-rotation of equation (15b) is restricted by equation (14). As a result, not only are the spin coefficients ρ, μ, τ, and π, properly weighted quantities, but so is the spin coefficient ε. However, I will present the variables in M in Newman-Penrose rather than GHP notation, at the expense of having to deal with some variables which are not properly weighted.

4. AUXILIARY QUANTITIES

The "solution" of the field equations depends only on the initial data P, L, $\mathrm{Re}\,\hat{\Psi}_2^{(1)}$, and $\Phi_{11}^{(4)}$, which are functions of the coordinates u, ζ, and $\bar{\zeta}'$ of J^+. But it is advisable, for the sake of brevity, to introduce some auxiliary quantities. Although all of these are quantities defined on J^+, some of them will not carry a caret, since they appear as coefficients in the expansion of some spacetime variable in terms of the spin coefficients ρ and $\bar{\rho}$.

First let us recall from Ludwig (1978) that we may define the operators $\hat{\delta}^0$, $\hat{\bar{\delta}}^0$, and $\hat{\Delta}^0$ and the spin coefficients $\hat{\tau}^0$, $\hat{\alpha}^0$, and $\hat{\mu}^0$ (and their conjugates) by

$$\hat{\delta}^0 = -L\frac{\partial}{\partial u} - 2P\frac{\partial}{\partial \zeta}$$
$$\hat{\Delta}^0 = \frac{\partial}{\partial \mathrm{u}}$$
$$\hat{\tau}^0 = P\hat{\Delta}^0\left(\frac{L}{P}\right) \tag{23}$$
$$\hat{\alpha}^0 = \tfrac{1}{2}\hat{\bar{\delta}}^0 \ln P$$
$$\hat{\mu}^0 = -\hat{\Delta}^0 \ln P$$

Although none of these are weighted quantities, the equations themselves are. These operators and spin coefficients play their major role in the definition of the operators $\hat{\delta}_c^0$, $\hat{\bar{\delta}}_c^0$, and $\hat{\Delta}_c^0$.

Next let us define

$$\begin{aligned}
2i\Sigma &= P\hat{\delta}^0\left(\frac{\bar{L}}{P}\right) - P\hat{\bar{\delta}}^0\left(\frac{L}{P}\right) \\
U^0 &= \mathrm{Re}[\hat{\delta}^0(\hat{\bar{\tau}}^0 - 2\hat{\alpha}^0) - 2\hat{\bar{\alpha}}^0(\hat{\bar{\tau}}^0 - 2\hat{\alpha}^0)] \\
\nu^0 &= \hat{\Delta}^0(\hat{\bar{\tau}}^0 - 2\hat{\alpha}^0) + \hat{\mu}^0(\hat{\bar{\tau}}^0 - 2\hat{\alpha}^0)
\end{aligned} \tag{24}$$

(and similarly for $\bar{\nu}^0$). These are properly weighted quantities with respective weights $(-1, 0)$, $(-2, 0)$, and $(-2, -1)$, as can be determined with the aid of equations (13), (16), and (23). Note that Σ is the imaginary part of $-\hat{\rho}$ and therefore the twist of the null geodesic congruence.

Further, we define the following auxiliary quantities and their complex conjugates in terms of properly weighted quantities defined on J^+. These auxiliary variables, which will appear as coefficients in the expansion of the variable whose name they carry, are themselves properly weighted. This is so since each term in their definition has the same weight. Table I gives a summary of these weights. For the conjugate variables the spin weight S changes to $-S$.

$$\begin{aligned}
\mathrm{Im}\,\hat{\Psi}_2^{(1)} &= \mathrm{Re}(-2\Sigma U^0 + \hat{\delta}_c^0\hat{\bar{\delta}}_c^0\Sigma) \\
\Psi_3^{(2)} &= -i\Sigma\nu^0 + \hat{\bar{\delta}}_c^0 U^0 + i\hat{\Delta}_c^0\hat{\bar{\delta}}_c^0\Sigma \\
\Psi_3^{(3)} &= \hat{\bar{\delta}}_c^0\hat{\Psi}_2^{(1)} \\
\Psi_3^{(4)} &= -3i\hat{\Psi}_2^{(1)}\hat{\bar{\delta}}_c^0\Sigma \\
\Phi_{12}^{(3)} &= \tfrac{1}{2}\hat{\delta}_c^0\hat{\Psi}_2^{(1)} \\
\Phi_{12}^{(4)} &= -\hat{\delta}_c^0\Phi_{11}^{(4)} \\
\Phi_{12}^{(5)} &= -2i(\hat{\delta}_c^0\Sigma)\Phi_{11}^{(4)} \\
\Psi_4^{(1)} &= -\hat{\bar{\delta}}_c^0\nu^0 \\
\Psi_4^{(2)} &= -\hat{\bar{\delta}}_c^0\Psi_3^{(2)} \\
\Psi_4^{(3)} &= -\hat{\bar{\delta}}_c^0\Psi_3^{(3)} + 4i\Psi_3^{(2)}\hat{\bar{\delta}}_c^0\Sigma \\
\Psi_4^{(4)} &= -\hat{\bar{\delta}}_c^0\Psi_3^{(4)} + 6i\Psi_3^{(3)}\hat{\bar{\delta}}_c^0\Sigma \\
\Psi_4^{(5)} &= 24\hat{\Psi}_2^{(1)}(\hat{\bar{\delta}}_c^0\Sigma)^2 \\
\Psi_4^{(a)} &= -\hat{\bar{\delta}}_c^0\Phi_{21}^{(3)} \\
\Psi_4^{(b)} &= -\hat{\bar{\delta}}_c^0\Phi_{21}^{(4)} + 2i\Phi_{21}^{(3)}\hat{\bar{\delta}}_c^0\Sigma \\
\Psi_4^{(c)} &= -\hat{\bar{\delta}}_c^0\Phi_{21}^{(5)} + 4i\Phi_{21}^{(4)}\hat{\bar{\delta}}_c^0\Sigma \\
\Psi_4^{(d)} &= 6i\Phi_{21}^{(5)}\hat{\bar{\delta}}_c^0\Sigma
\end{aligned} \tag{25}$$

Table I

Quantity	W	S	Quantity	W	S
Ω	1	0	$\Psi_3^{(4)}, \Phi_{21}^{(4)}$	−5	−1
$\hat{\Delta}_c^0, \Sigma$	−1	0	$\Phi_{21}^{(5)}$	−6	−1
U^0	−2	0	$\Psi_4^{(1)}$	−3	−2
$\hat{\Psi}_2^{(1)}$	−3	0	$\Psi_4^{(2)}$	−4	−2
$\Phi_{11}^{(4)}$	−4	0	$\Psi_4^{(3)}, \Psi_4^{(a)}$	−5	−2
$\hat{\delta}_c^0$	−1	−1	$\Psi_4^{(4)}, \Psi_4^{(b)}$	−6	−2
ν^0	−2	−1	$\Psi_4^{(5)}, \Psi_4^{(c)}$	−7	−2
$\Psi_3^{(2)}$	−3	−1	$\Psi_4^{(d)}$	−8	−2
$\Psi_3^{(3)}, \Phi_{21}^{(3)}$	−4	−1			

5. THE "SOLUTION"

One can exhibit the "solution" of the field equations for algebraically special spacetimes in a rather concise manner in terms of these auxiliary quantities, which themselves depend only on the initial data, defined on J^+, and their derivatives tangential to J^+. Although this "solution" has been obtained elsewhere (Trim and Wainwright, 1974; Ludwig, 1978), it takes considerable effort to transcribe it into the present formalism. The results are as follows.

Spin Coefficients:

$$\begin{aligned}
&0=\kappa=\sigma=\varepsilon=\pi=\tau=\lambda\\
&\rho=-\frac{1}{r-i\Sigma}\\
&\alpha=\rho(\hat{\bar{\tau}}^0-\hat{\alpha}^0)\\
&\beta=\bar{\rho}\hat{\bar{\alpha}}^0\\
&\gamma=-\tfrac{1}{2}\hat{\Delta}^0\ln P-\tfrac{1}{2}\rho^2\hat{\Psi}_2^{(1)}+\rho^2\bar{\rho}\Phi_{11}^{(4)}\\
&\mu=-\tfrac{1}{2}\hat{\Psi}_2^{(1)}(\rho^2+\rho\bar{\rho})+\Phi_{11}^{(4)}\rho^2\bar{\rho}-\bar{\rho}[U^0+i\hat{\Delta}_c^0\Sigma]\\
&\nu=\nu^0+\rho\Psi_3^{(2)}+\tfrac{1}{2}\rho^2\Psi_3^{(3)}+\tfrac{1}{3}\rho^3\Psi_3^{(4)}+\rho\bar{\rho}\Phi_{21}^{(3)}+\rho^2\bar{\rho}\Phi_{21}^{(4)}+\rho^3\bar{\rho}\Phi_{21}^{(5)}
\end{aligned}\tag{26}$$

Weyl Tensor Components:

$$\begin{aligned}
&\Psi_0=\Psi_1=0\\
&\Psi_2=-\hat{\Psi}_2^{(1)}\rho^3+2\Phi_{11}^{(4)}\rho^3\bar{\rho}\\
&\Psi_3=\Psi_3^{(2)}\rho^2+\Psi_3^{(3)}\rho^3+\Psi_3^{(4)}\rho^4+\Phi_{21}^{(3)}\rho^2\bar{\rho}+2\rho^3\bar{\rho}\Phi_{21}^{(4)}+3\rho^4\bar{\rho}\Phi_{21}^{(5)}\\
&\Psi_4=\Psi_4^{(1)}\rho+\Psi_4^{(2)}\rho^2+\tfrac{1}{2}\Psi_4^{(3)}\rho^3+\tfrac{1}{3}\Psi_4^{(4)}\rho^4+\tfrac{1}{4}\Psi_4^{(5)}\rho^5\\
&\qquad+\bar{\rho}\rho^2\Psi_4^a+\bar{\rho}\rho^3\Psi_4^b+\bar{\rho}\rho^4\Psi_4^c+\bar{\rho}\rho^5\Psi_4^d
\end{aligned}\tag{27}$$

Ricci Tensor Components:

$$\begin{aligned}
&\Lambda = \Phi_{00} = \Phi_{01} = \Phi_{02} = 0\\
&\Phi_{11} = \Phi_{11}^{(4)}\rho^2\bar{\rho}^2\\
&\Phi_{12} = \rho^2\bar{\rho}\Phi_{12}^{(3)} + \rho^2\bar{\rho}^2\Phi_{12}^{(4)} + \rho^2\bar{\rho}^3\Phi_{12}^{(5)}\\
&\Phi_{22} = \rho\bar{\rho}[-\hat{\delta}_c^0\Psi_3^{(2)} + \hat{\Delta}_c^0\hat{\Psi}_2^{(1)}] - \rho^2\bar{\rho}[\hat{\bar{\delta}}_c^0\Phi_{12}^{(3)} + \tfrac{1}{2}\hat{\Delta}_c^0\Phi_{11}^{(4)}] + \rho^3\bar{\rho}[2i\Phi_{12}^{(3)}\hat{\bar{\delta}}_c^0\Sigma]\\
&\qquad - \rho\bar{\rho}^2[\hat{\delta}_c^0\Phi_{21}^{(3)} + \tfrac{1}{2}\hat{\Delta}_c^0\Phi_{11}^{(4)}] - \rho^2\bar{\rho}^2 \,\mathrm{Re}\, \hat{\delta}_c^0\Phi_{21}^{(4)} + \rho^3\bar{\rho}^2[2i\Phi_{12}^{(4)}\hat{\bar{\delta}}_c^0\Sigma]\\
&\qquad - \rho\bar{\rho}^3[2i\Phi_{21}^{(3)}\hat{\delta}_c^0\Sigma] - \rho^2\bar{\rho}^3[2i\Phi_{21}^{(4)}\hat{\delta}_c^0\Sigma] + \rho^3\bar{\rho}^3[4\Phi_{11}^{(4)}|\hat{\delta}_c^0\Sigma|^2]
\end{aligned} \tag{28}$$

Metric Variables:

$$\begin{aligned}
X &= \xi_2 = 0\\
\xi_0 &= L\bar{\rho}\\
\xi_1 &= 2P\bar{\rho}\\
\omega &= \hat{\bar{\tau}}^0[1 + i\Sigma\bar{\rho}] - i\bar{\rho}\hat{\delta}_c^0\Sigma\\
U &= -r\hat{\mu}^0 + U^0 + \tfrac{1}{2}\rho\hat{\Psi}_2^{(1)} + \tfrac{1}{2}\bar{\rho}\hat{\bar{\Psi}}_2^{(1)} - \rho\bar{\rho}\Phi_{11}^{(4)}
\end{aligned} \tag{29}$$

Differential Operators and Tetrad:

$$\begin{aligned}
D &= \frac{\partial}{\partial r}\\
\delta &= \xi_0\frac{\partial}{\partial u} + \omega\frac{\partial}{\partial r} + \xi_1\frac{\partial}{\partial \zeta}\\
\Delta &= \frac{\partial}{\partial u} + U\frac{\partial}{\partial r}\\
k^a &= (0, 1, 0, 0)\\
n^a &= (1, U, 0, 0)\\
m^a &= (\xi_0, \omega, \xi_1, 0)\\
k_a &= \left(1, 0, -\frac{L}{2P}, -\frac{\bar{L}}{2P}\right)\\
m_a &= \left(0, 0, 0, -\frac{1}{2P\rho}\right)\\
n_a &= \left(-U, 1, \frac{1}{2P}\left(UL - \frac{\omega}{\bar{\rho}}\right), \frac{1}{2P}\left(U\bar{L} - \frac{\bar{\omega}}{\rho}\right)\right)
\end{aligned} \tag{30}$$

with respect to the coordinates $(u, r, \zeta, \bar{\zeta})$.

Line Element:

$$ds^2 = -(2P^2\rho\bar{\rho})^{-1}\, d\zeta\, d\bar{\zeta} + 2(k_a\, dx^a)(n_b\, dx^b)$$

where

$$k_a\, dx^a = du - \frac{L}{2P}\, d\zeta - \frac{\bar{L}}{2P}\, d\bar{\zeta}$$
$$n_b\, dx^b = -Uk_a\, dx^a + dr - \frac{\omega}{2P\bar{\rho}}\, d\zeta - \frac{\bar{\omega}}{2P\rho}\, d\bar{\zeta} \qquad (31)$$

6. DISCUSSION

It is now a simple matter to verify the form invariance of this "solution" under the remaining freedom in the choice of coordinate system and tetrad. Let us first note that we can easily calculate from their definitions that under such a permissible transformation, given by equations (13)-(16), we have

$$X' = \Theta^{-1}\, e^{-2i\phi} \left|\frac{d\zeta'}{d\zeta}\right| X$$

$$\xi_{0'} - L'\bar{\rho}' = e^{2i\phi}\left[\Theta(\xi_0 - \bar{\rho}L) + \frac{\partial G}{\partial\zeta}(\xi_1 - 2P\bar{\rho})\right]$$

$$\xi_1' - 2P'\bar{\rho}' = \left|\frac{d\zeta'}{d\zeta}\right|(\xi_1 - 2P\bar{\rho})$$

$$\xi_2' = e^{4i\phi}\left|\frac{d\zeta'}{d\zeta}\right| \xi_2$$

$$\omega' - \hat{\tau}^{0'}(1 + i\Sigma'\bar{\rho}') = \Theta^{-1}\, e^{2i\phi}\left[\omega - \hat{\tau}^0(1 + i\Sigma\bar{\rho}) - r(\xi_0 - L\bar{\rho})\frac{\partial \ln\Theta}{\partial u}\right.$$
$$\left. - r(\xi_1 - 2P\bar{\rho})\frac{\partial \ln\Theta}{\partial\zeta}\right] \qquad (32)$$

This shows the invariance of most equations in (29). To show that all of equations (26)-(31) are invariant, we simply verify that each of the remaining equations is weighted. A list of the weights involved is given in Table II. These weights can be either calculated directly from known transformation rules of the Newman-Penrose variables or transcribed from the appropriate table in Ludwig (1988) using equation (22).

The results of the last section turn into an actual solution of the field equations once we solve the "reduced field equations" for the initial data given by (12). The latter equations are obtained by putting the expressions

Table II

Variable	W	S	Variable	W	S
r	−1	0	Ψ_3	−1	−1
κ	2	1	Ψ_4	−2	−2
σ	1	2	Λ	0	0
π	0	−1	Φ_{00}	2	0
ε	1	0	Φ_{01}	1	1
τ	0	1	Φ_{02}	0	2
λ	−1	−2	Φ_{11}	0	0
ρ	1	0	Φ_{12}	−1	1
μ	−1	0	Φ_{22}	−2	0
ν	−2	−1	$\gamma-\frac{1}{2}\hat{\mu}^0$	−1	0
Ψ_0	2	2	$\beta-\bar{\rho}\hat{\bar{\alpha}}^0$	0	1
Ψ_1	1	1	$\alpha-\rho(\hat{\bar{\tau}}^0-\hat{\alpha}^0)$	0	−1
Ψ_2	0	0	$U+r\hat{\mu}^0$	−2	0

for Φ_{11}, Φ_{12}, and Φ_{22} of (28) equal to an appropriate source. For example, the reduced equations for vacuum are

$$\Phi_{11}^{(4)}=0, \qquad \hat{\delta}_c^0\hat{\Psi}_2^{(1)}=0, \qquad \hat{\delta}_c^0\Psi_3^{(2)}-\hat{\Delta}_c^0\hat{\Psi}_2^{(1)}=0 \tag{33}$$

Further conditions on the initial data may arise as a result of insisting on a specific Petrov type or on some other geometrical property of the solution.

Initial data for the Kerr-Newman solution are

$$P=\frac{1}{2\sqrt{2}}(1+\zeta\bar{\zeta}), \qquad L=-\frac{ia}{2P}\bar{\zeta}, \qquad \operatorname{Re}\hat{\Psi}_2^{(1)}=-m, \qquad \Phi_{11}^{(4)}=\tfrac{1}{2}e^2 \tag{34}$$

where a, m, and e are constants. For the NUT solution we can take

$$P=\frac{1}{2\sqrt{2}}(1+\zeta\bar{\zeta}), \qquad L=\frac{ia}{\sqrt{2}}\left(\bar{\zeta}-\frac{1}{\zeta}\right), \qquad \operatorname{Re}\hat{\Psi}_2^{(1)}=-m, \qquad \Phi_{11}^{(4)}=0 \tag{35}$$

where a and m are constants. The actual solutions may now be written down immediately by substituting these initial data into (26)-(31). In particular, the respective line elements are obtained from (31).

The formalism can be readily generalized to complex spacetimes by letting real quantities take on complex values and by letting a variable and its complex conjugate become independent. This complexification will allow us to reexamine, from a different point of view, the complex coordinate trick of Newman and Janis (1965) that yields the Kerr solution from the Schwarzschild and Demianski (1972) generalization thereof. This will be done elsewhere.

ACKNOWLEDGMENT

I am indebted to the Natural Science and Engineering Research Council of Canada for its financial support.

REFERENCES

Debney, G. C., Kerr, R. P., and Schild, A. (1969). *Journal of Mathematical Physics*, **10**, 1842-1854.

Demianski, M. (1972). *Physics Letters*, **42A**, 157-159.

Geroch, R., Held, A., and Penrose, R. (1973). *Journal of Mathematical Physics*, **14**, 874-881.

Kerr, R. P. (1963). *Physical Review Letters*, **11**, 237-238.

Lind, R. W. (1974). *General Relativity and Gravitation*, **5**, 25-47.

Ludwig, G. (1976). *General Relativity and Gravitation*, **7**, 293-311.

Ludwig, G. (1978). *General Relativity and Gravitation*, **9**, 1009-1019.

Ludwig, G. (1986). *Classical and Quantum Gravity*, **3**, L141-L147.

Ludwig, G. (1988). *International Journal of Theoretical Physics*, **27**, 315-333.

Newman, E. T., and Janis, A. I. (1965). *Journal of Mathematical Physics*, **6**, 915-917.

Newman, E., and Penrose, R. (1962). *Journal of Mathematical Physics*, **3**, 566-578.

Penrose, R. (1968). In *Battelle Rencontres* (C. de Witt and J. A. Wheeler, eds.), W. A. Benjamin, New York.

Pirani, F. A. E. (1965). In *Lectures on General Relativity*, pp. 350-351, Prentice-Hall, Englewood Cliffs, New Jersey.

Robinson, I., Robinson, J., and Zund, J. D. (1969). *Journal of Mathematics and Mechanics*, **18**, 881-892.

Talbot, C. J. (1969). *Communications in Mathematical Physics*, **13**, 45-61.

Trim, D. W., and Wainwright, J. (1974). *Journal of Mathematical Physics*, **15**, 535-546.